VECTOR SPACES

1 Intersection of linear subspaces

Proposition (Intersection of linear subspace).

Let E be a vector space over $\mathbb{R}$, F and G two linear subspaces of E. Then $F \cap G$ is a linear subspace of E.

Proof:

- $F \subset E$ and $G \subset E \Longrightarrow F \cap G \subset E$.
- F and G linear subspaces of $E \Longrightarrow 0_E \in F$ and $0_E \in G \Longrightarrow 0_E \in F \cap G$.
- Let $(u, v) \in (F \cap G)^2$ and $\alpha \in \mathbb{R}$. Then, $(u, v) \in F^2$ and $(u, v) \in G^2$. Thus,

$$F \text{ linear subspace of } E \Longrightarrow \alpha\, u + v \in F.$$
$$G \text{ linear subspace of } E \Longrightarrow \alpha\, u + v \in G.$$

Finally, $\alpha\, u + v \in F \cap G$.

2 Sum of linear subspaces

Proposition (Sum of linear subspaces).

Let E be a vector space over $\mathbb{R}$, F and G two linear subspaces of E. Then $F + G$ is a linear subspace of E.

Proof:

Remind that $F + G = \{u \in E, \exists\, (u_1, u_2) \in F \times G, u = u_1 + u_2\}$.

Thus, a vector u of E is in $F + G$ if and only if it can be written as the sum of a F-vector and a G-vector:

$$u = \underbrace{u_1}_{\in F} + \underbrace{u_2}_{\in G}$$

- By definition, $F + G \subset E$.
- $0_E = \underbrace{0_E}_{\in F} + \underbrace{0_E}_{\in G}$ because F and G are linear subspaces of E. Hence, $0_E \in F + G$.
- Let $(u, v) \in (F + G)^2$ and $\alpha \in \mathbb{R}$.

$\exists\, (u_1, u_2) \in F \times G$ and $\exists (v_1, v_2) \in F \times G$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$. Thus,

$$\alpha\, u + v = \alpha\, (u_1 + u_2) + (v_1 + v_2) = \underbrace{(\alpha\, u_1 + v_1)}_{\in F} + \underbrace{(\alpha\, u_2 + v_2)}_{\in G}$$

since F and G are linear subspaces of E. Hence, $\alpha\, u + v \in F + G$.

Theorem (**Uniqueness of the decomposition in $F+G$**).

Let E be a vector space over $\mathbb{R}$, F and G two linear subspaces of E in direct sum, that is, $F \cap G = \{0_E\}$.
Then the decomposition in $F+G$ is unique:

$$\forall\, u \in F+G, \exists\,!(u_1,u_2) \in F \times G,\ u = u_1 + u_2$$

Proof:

Let $u \in F+G$. Assume there are two ways of decomposing u:

$$u = u_1 + u_2 \ \text{ with } \ (u_1,u_2) \in F \times G \ \text{ and } \ u = v_1 + v_2 \ \text{ with } \ (v_1,v_2) \in F \times G$$

Then, $u_1 + u_2 = v_1 + v_2 \Longrightarrow u_1 - v_1 = v_2 - u_2$.

Since $(u_1, v_1) \in F^2$ and F is a linear subspace of E, $u_1 - v_1 \in F$. Similarly, $v_2 - u_2 \in G$ since $u_2 \in G$, $v_2 \in G$ and G is a linear subspace of E.

Thus, $\underbrace{u_1 - v_1}_{\in F} = \underbrace{v_2 - u_2}_{\in G}$

We hence get: $u_1 - v_1 \in F \cap G$. Since F and G are in direct sum ($F \cap G = \{0_E\}$), we deduce that $u_1 - v_1 = 0_E$, that is, $u_1 = v_1$.
By injecting this in: $u = u_1 + u_2 = v_1 + v_2$, we get: $u_2 = v_2$. The decomposition is hence unique.

3 Spanned linear subspaces

Proposition (**Spanned linear subspaces**).

Let E be a vector space over $\mathbb{R}$ and $(u_1, u_2, \ldots, u_n) \in E^n$. If u_n is a linear combination of $u_1, \ldots, u_{n-1}$, then

$$\mathit{Span}\,((u_1,\ldots,u_{n-1},u_n)) = \mathit{Span}\,((u_1,\ldots,u_{n-1}))$$

Proof:

$\boxed{\subset}$ Let $v \in \text{Span}\,((u_1,\ldots,u_{n-1},u_n))$.

$$\exists\,(\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n \text{ such that } v = \alpha_1 u_1 + \ldots + \alpha_{n-1}u_{n-1} + \alpha_n u_n$$

Furthermore, we assume that $\exists\,(\lambda_1,\ldots,\lambda_{n-1}) \in \mathbb{R}^{n-1}$ such that $u_n = \lambda_1 u_1 + \ldots + \lambda_{n-1}u_{n-1}$.

This relation leads to:

$$\begin{aligned} v &= \alpha_1 u_1 + \ldots + \alpha_{n-1}u_{n-1} + \alpha_n(\lambda_1 u_1 + \cdots + \lambda_{n-1}u_{n-1}) \\ &= (\alpha_1 + \alpha_n\lambda_1)u_1 + \ldots + (\alpha_{n-1} + \alpha_n\lambda_{n-1})u_{n-1} \end{aligned}$$

Hence, $v \in \text{Span}\,((u_1,\ldots,u_{n-1}))$.

$\boxed{\supset}$ Let $v \in \text{Span}\,((u_1,\ldots,u_{n-1}))$. Then

$$\begin{aligned} \exists\,(\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{R}^{n-1} \text{ such that } \ v &= \alpha_1 u_1 + \ldots + \alpha_{n-1}u_{n-1} \\ &= \alpha_1 u_1 + \ldots + \alpha_{n-1}u_{n-1} + 0u_n \end{aligned}$$

Hence, $v \in \text{Span}\,((u_1,\ldots,u_{n-1},u_n))$.